Computing Three Dimensional Fluids

Dennis Sullivan

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In order to compute fluid motion from a continuum fluid model or a physical picture, it must be discretized in terms of finitely many parameters. Discretizing space by dividing it into cells was Poincaré's starting point when he began topology to study qualitative dynamical systems just over one hundred years ago. In the middle of the twentieth century, great advances were made in algebraic aspects of topology, which is also based on those cells of Poincaré. These advances are actually related to algebraic products that are involved in the discretization process for the nonlinear term of the fluid models.

Discretizing continuum models in any manner must break some of the algebraic symmetry present. This breaking of symmetry can be theoretically treated by a hierarchy of corrections based on algebraic topology. These breaks or discrepancies are the cumulants of statistics which are expressed via algebraic topology as integrals of differentials in the sense of calculus. They are also related to the Feynman diagrams used in the algorithms to compute physical effects in quantum theories.

Sullivan and his student colleagues have been engaged in understanding these corrections and building theoretical machinery for fluid computations based on these ideas. The current work led to the revelation/understanding that different ways of discretizing various rewritings of the continuum model which are equivalent at the continuum level can be inequivalent at the discrete level because of this broken symmetry.

There is a theoretical consistency, however, if one allows for the extended sequence of corrections alluded to above. Systematically testing fluid data against the various algorithms in terms of these extended corrections would be beneficial. With the second part of his Balzan Prize, Sullivan has initiated testing the practical aspects of this theoretical work.

The first construction of an algorithm for computing incompressible fluid motion, the momentum model, which replaces the continuum language by that of combinatorial topology and a discrete lattice multivector algebra, has been completed. One interesting point is that the algorithm is first derived from a tautologous conservation principle on a face centered-cubic lattice in three dimensional space, which is an eight-layered covering by all cubical cells of edge length twice the lattice scale. This first algorithm is NOT derived from the continuum model, but derived directly from momentum preservation expressed at the discrete level. Therefore, this first algorithm tends as the scale tends to zero to the continuum model, which is also derived from momentum preservation in a calculus limit. The discrete algorithm, after a closure type cumulant discard appears symbolically written exactly in Leray's form, with the derivative outside the nonlinear terms of the discrete multivector calculus notation.

This Leray form at the continuum level allows the concept and precise definition of generalized or weak solutions. The idea first tested was to imagine the algorithm to be written at such a small scale (above the atomic scale) that the momentum vectors on each face are essentially constant (so that the cumulant discard is justified). Then the lattice and algorithm are amalgamated as in Wilson renormalization to reach a coarser level where the computer budget makes computation feasible. A description of this algorithm will appear in the memorial volume for Jean-Christophe Yoccoz (Collège de France) entitled *Lattice Hydrodynamics* (2020). See also Dennis Sullivan's presentation of *Lattice Hydrodynamics* at The Simons Center for Geometry and Physics Video Portal (http://scgp.stonybrook.edu/video_portal/).

Recently, with Nissim Ranade and Ruth Lawrence, a more penetrating analysis was made of the hierarchy of finite dimensional discrete vector calculus with estimates on the cumulant discrepancies of the broken symmetries of the continuum model under this FCC discretization. The result was quite powerful and positive, showing that the estimates were there to justify perturbation theory as it is used in the Feynmann diagrams of perturbative quantum field theory. This will appear in the Sir Michael Atiyah Memorial Volume Quarterly Journal of Mathematics 2021.

This positive news dovetails with the following definite negative results. Sullivan and his group have discovered that the first algorithm is unstable numerically, and within current computational budgets can be stabilized by the up down procedure indicated above only up to Reynolds number about 1000. We think this is a significant finding, because it shows a naturally derived algorithm is not practical beyond reasonably smooth situations.

In response to this result, the focus is now on the second model, the vorticity model, which descibes fluid motion in terms of vorticity evolution verbally: by first transporting vorticity then diffusing it isotropically. This model is also described in the Yoccoz and Atiyah volumes noted above.

Written in the discrete vector calculus this form is exactly the familiar Navier Stokes model with the derivative inside the nonlinear term. The first simulations show good numerical stability even for quite small values of viscosity. The current goal is to test this stability more fully with super computer simulation and a broader set of initial conditions. If that stability persists, results will be compared with accepted data, both simulated and experimental.

Sullivan's Balzan research project is primarily based at Stony Brook University, with parts being carried out at the Graduate Center of the City University of New York.

A conference, "Real Fluids in dim ≤ 3 | Complex Manifolds in dim ≥ 3", was held at the CUNY Graduate Center in April 2018. For further details, see Simons-mathfest2018.ws.gc.cuny.edu.