wonderful places such as Texas. I am really happy to see that a single mathematician in Brazil can generate gigantic waves all around the world, significantly expanding our philosophical and mathematical view of the world around us.

Jacob Palis:

Dynamical Systems. Chaotic Behaviour – Uncertainty

Thank you, Etienne Ghys for such kind words. I would also like to thank the Balzan Foundation and the General Prize Committee for bestowing upon me this great honour and at the same time the Accademia Nazionale dei Lincei for recently conferring upon me membership.

I am very happy to be here with my family, including my wife, my children and my 6-year-old grandson.

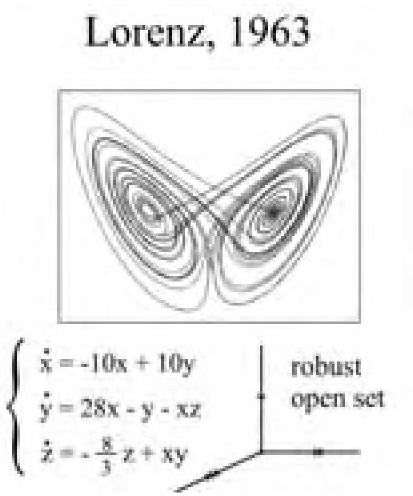
Despite the fact that I will deal with Chaos and Uncertainty, my lecture will contain a positive message.

Although the theory of dynamics can be traced back to the Greeks or even beyond, we attribute to the great French mathematician Henri Poincaré the creation, in the second half of the nineteenth century, of the modern theory of dynamical systems.

Here we understand dynamical systems as flows or transformations in a space of events, or phase space, that we take to be a compact surface of any dimension without boundary. We shall assume that the flows and the transformations are at least C^1, that is, continuously differentiable and that the transformations are inverse with the same such property, that is, they are diffeomorphisms. In general, we say that the dynamical system, diffeomorphism or flow, is of class C^r, r a positive integer, if it is r-continuously differentiable. Such systems represent one of the main mathematical instruments used to model the evolution of many phenomena in nature and, more broadly, in other areas of science, through transformations of a space of events into itself or through differential equations that generate flows on the phase space.

Classical examples are population growth of species and weather and climate prediction; perhaps the same theory can be applied to understand certain aspects of turbulence and other important phenomena. From a set of initial data, or a point in the space of events, we apply the model many times in the case of transformations or for a long time when using differential equations: in both cases, we name trajectories the set of iterates from an initial point. If, starting at two very close initial points, behaviour of the trajectories over a long timeframe vary substantially, we call this system chaotic. This is the case for many systems, and in fact uncertainty is very common in phenomena in nature and beyond.

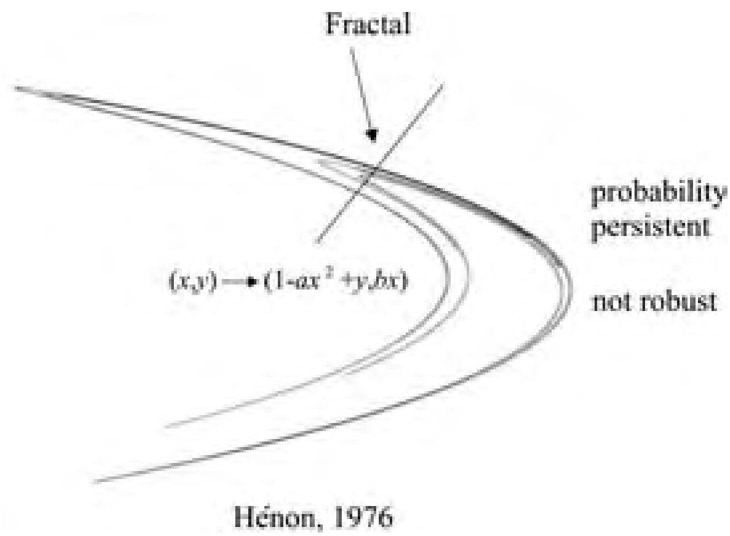
The well-known 1963 Lorenz model for weather prediction consists of a flow in three dimensions displaying an invariant butterfly-like figure that attracts all trajectories starting at points near it – the so called Lorenz butterfly-like attractor. Moreover, if we consider very close initial points, it is, very likely that, after a long period of time, the trajectories will be further apart as much as the maximum distance between any two points in the butterfly figure, that is, the diameter of the butterfly figure. Thus, we have uncertainty about the future behaviour of the trajectories, which in this case is measured by the diameter of the butterfly. Dynamical systems with similar behaviour are called chaotic.



Moreover, the Lorenz model is robust from a mathematical point of view: if we change slightly the numbers (coefficients) that appear in the equations of the model, we still have a similar butterfly-like figure attracting the future trajectories starting at nearby points. See reference [BDV]. To be chaotic is not such a rare phenomenon among dynamical systems: the exception is when the attracting figures for the future trajectories consist of a finite number of points or periodic trajectories.

Concerning attractors other than hyperbolic ones, two of them have been most remarkable and they have influenced the development of dynamics from the 1970s on.

The first one is the Lorenz attractor, that we have just briefly commented on, the other one is due to Hénon. It should be added that, like Hénon, Jakobson exhibited new attractors for parametrized families of unimodal maps of the interval that are probability persistent, but not robust, under a small perturbation of the parameter. Benedicks and Carleson showed the existence of Hénon-like attractors. Making use of their results, Mora and Viana proved that they appear in homoclinic bifurcations.



Curiously, many dynamicists were not aware of Lorenz's work when I first came to the University of California at Berkeley, in 1964, after graduating from the School of Engineering of the University of Brazil, now known as the Federal University of Rio de Janeiro. The main focus was on hyperbolic systems, as I will discuss in the following text, and they did not include the robust case of the Lorenz attractor.

My first contribution to the area started precisely with dynamical systems displaying a discrete set of fixed or periodic orbits, or more precisely with the so-called Morse-Smale systems. Smale was my adviser at the University of California at Berkeley. At the time, much emphasis was given to the set of hyperbolic systems, and the Morse-Smale systems were part of such a set. In simple terms, a system is hyperbolic if along its trajectories distances increase and decrease exponentially in complementary dimensions, or in complementary dimensions transversally to flow trajectories. More formally, let M be the phase space and f a transformation on it. A point x in M is called nonwandering if in any neighbourhood V of x, we can find a point y such that $f^n(y)$ is in V for some integer n. The union of all nonwandering points is called the nonwandering set of f, $\Omega(f)$. Similarly for flows. Indeed, we say that f or the flow X_t, t a real number, is hyperbolic if $\Omega(f)$, or $\Omega(X_t)$, is hyperbolic. Weaker, but still very relevant, are the concepts of partial hyperbolicity and dominated splitting that again describes the relative growth of distances along trajectories in complementary or sub-complementary dimensions. It is suggested that the reader consult (BDV) for a comprehensive presentation of basic concepts and definitions that appear here.

Under the hyperbolicity hypothesis, through each point x in Ω, we have a stable line or plane formed by points whose trajectories follow and approach the trajectory of x, which is called the stable manifold of x. There is a dual concept of unstable manifold. Also for flows we can give similar definitions. If $\Omega(f)$ is hyperbolic and the periodic points of f are dense in $\Omega(f)$, we say that f satisfies Axiom A. Finally, we can impose the transversality condition: for every pair x, y of points in $\Omega(f)$ the stable manifold of x is transversal to the unstable manifold of y. In the particular case that $\Omega(f)$ is made of a finite number of fixed or periodic orbits, we call f a Morse-Smale diffeomorphism. Similarly for a Morse-Smale flow.

In my PhD thesis in 1967, I proved that Morse-Smale systems, diffeomorphisms or flows, are structurally stable in low dimensions, up to and including three. That is, given a Morse-Smale diffeomorphism f, for any g r-differentiably close to f (C^r close, $r \geq 1$), there is a continuous one to one transformation h of M, such that $h(f(x)) = g(h(x))$. From that, it follows that $h(f^n(x)) = g^n(h(x))$, that is, h sends orbits of f into orbits of g. In other words, the orbit structure of Morse-Smale diffeomorphisms or flows are unchanged by small perturbations of the systems: they are structurally stable.

Earlier work by Andronov-Pontryagin and Peixoto considered the case of flows on a disc and on surfaces, respectively.

I had to go beyond that to treat the case of diffeomorphisms in two and three dimensions and flows in this latter case. To do so, I have created the notion of stable foliations being partially subfoliated to include the ones of fixed or periodic points of higher indices where they accumulate upon. This notion has fundamentally influenced the subsequent development in this line of research.

Right after that, in a joint work with Smale, the results of my thesis were extended to all dimensions.

We also formulated conjectures that became quite famous, namely the Stability Conjectures, that proposed precise conditions for a dynamical system to be structurally stable or stable restricted to the nonwandering set. These conjectures were a major topic of research in the area for more than two decades.

To formulate them, we note that when f satisfies Axiom A, $\Omega(f)$ splits into a finite number of disjoint (compact) pieces Ω_i, $1 \leq i \leq n$, each of them displaying a dense orbit, i.e., it is transitive, as proved by Smale. They are called basic sets for f. A k-cycle on $\Omega = \Omega(f)$, $k \geq 1$, is a sequence of basic sets $\Omega_0, \Omega_1, \ldots, \Omega_{k+1} = \Omega_0$ (re-ordering indices if necessary) and points $x_1, x_2, \ldots, x_k$ outside Ω, such that the negative orbit of x_i accumulates on Ω_{i-1} and its positive orbit on Ω_i.

The Stability Conjectures state that a C_r diffeomorphisms f is structurally stable if and only if $\Omega(f)$ is hyperbolic and the transversality condition holds, and f is Ω-stable if and only if $\Omega(f)$ is hyperbolic with no cycles.

I proved that the non-cycle condition was necessary for Ω-stability of f, if $\Omega(f)$ is hyperbolic. This helped motivating the conjectures on stability. Besides the case of Morse-Smale systems dealt with by Palis-Smale, other fundamental contributions were given by Anosov in the 1960s and in the early 1970s by Robbin, Robinson and de Melo in his PhD thesis, the first one under my supervision at IMPA. A final solution in the C^1 category for diffeomorphisms was brilliantly obtained by Mañé, also my former PhD student in the early 1970s. I have completed the work of Mañé in the case of the stability restricted to the nonwandering set. The case of flows was later treated by Hayashi following Mañé's work.

I should say that I still remember with some emotion my first encounter with René Thom, a great French mathematician and a Fields Medalist like Steve Smale. It was in a meeting in Seattle in August of 1967. He was very interested in the proof of the structural stability of Morse-Smale systems, including the new ideas used to do so.

All the works above were published in outstanding journals such as the Annals of Mathematics, Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques, Inventiones Mathematicae, Global Analysis - American Mathematical Society.

After the conclusion of my PhD degree in September 1967, I visited a number of institutions on the East Coast of the United States, especially Brown University, and also MIT, for about six months. Then I returned to Berkeley for another period of six months as an Assistant Professor. At this point, I started shifting somewhat my interest in dynamics in terms of strategy, motivated by reading more intensively the work of Poincaré.

A strong motivation to go back to Berkeley was to once again enjoy its scientific atmosphere, although I had already decided to soon return to Brazil. Also, I wanted to participate in an important meeting to take place there in July 1968 on Global Analysis, including Dynamical Systems. S.S. Chern, a famous geometer, and Steve Smale would be the scientific coordinators.

I was invited to speak twice on my joint work with Smale, which I took as a special and very stimulating sign of appreciation. And I had a chance to become acquainted with many outstanding mathematicians, among them Jürgen Moser, with whom I would interact for the next three decades, most particularly at ETH-Zurich.

On the one hand, I continued to work on the hyperbolic theory of dynamics, as in my joint work with Hirsch, Pugh and Shub on stable manifolds of basic sets, published in Inventiones Mathematicae in 1971 and with Newhouse, published by Academic Press, a volume edited by Peixoto, in 1973.

On the other, I kept in mind the major challenge of providing a description of the "typical" behaviour of trajectories of a "typical" dynamical system.

Let me say that Poincaré was perhaps the first to set out in this direction. A serious attempt was made by Smale in the early sixties, when he introduced and proposed the hyperbolic systems as "typical". However, a few years later himself and Abraham provided a counter-example, followed by many others.

I was particularly impressed in 1968 with Newhouse's demonstration that hyperbolicity is not dense even on the two dimensional sphere for twice differentiable dynamical systems. He discovered it through a bifurcation of a Poincaré cycle, in this case of a homoclinic tangency, which we shall again discuss later.

Some twenty-five years later Newhouse's result was remarkably generalized to all dimensions by Viana and myself. I shall go back to this point later.

While I maintained significant interest in stability of orbit structure (structural stability), I moved to bifurcation theory, which concerns changes in orbit structure of systems depending on one or more parameters.

Following this line of research, I published, with Newhouse, "Bifurcations of Morse-Smale Systems" in 1973 in the volume by Academic Press mentioned above and "Cycles and Bifurcation Theory" in 1976 in Astérisque, vol. 31. We have shown the existence of simple bifurcations, as well as the ones leading to infinitely many different types of orbit structure related to the creation and unfolding of cycles. We have considerably extended Sotomayor's work for flows on two-dimensional surfaces.

To top all this off, Newhouse, myself and Takens published a long article in Publications Mathématiques de l'IHES, vol. 57. It may be considered a classic in this line of research. We have characterized stable arcs of dynamical systems whose limit sets consist of finitely many orbits. Models for the unfolding of the bifurcating periodic orbits were established, as well as moduli of stability related to saddle-connections and several results on the dynamical structure of the bifurcating diffeomorphisms.

I had a brief, but very interesting, incursion into holomorphic dynamics in a paper written jointly with Camacho and Kuiper, which appeared in Publications Mathématiques de l'IHES, vol. 48. It started with some differentiable invariants of topological conjugacies that I exposed during my invited discussion at the International Congress of Mathematicians in Helsinki, 1978. It helped the authors to classify linear holomorphic flows. The International Congresses of Mathematicians are held every four years by the International Mathematical Union. It is an important distinction to be invited to speak at one of them. The title of my talk was "Moduli of Stability and Bifurcation Theory".

By the end of 1970s, Wellington de Melo and myself wrote an introductory book called *Geometric Theory of Dynamical Systems*. It first appeared in Portuguese as Notes of the Brazilian Mathematical Colloquium and then in the collection Projeto Euclides of the Institute for Pure and Applied Mathematics - IMPA. It was translated into English by A. Manning and published by Springer Verlag. It was translated into Russian under the direction of D. Anosov and later also into Chinese. These became standard university texts worldwide and we are constantly inundated with requests for new editions.

Continuing with my research contribution to Dynamical Systems, I still pursued with Takens and later Dias Carneiro a classic question by René Thom, on the stability and bifurcation of parametrized families of gradient vector fields for one or more parameters. This was a charming question but a very difficult one, with a complicated interplay between dynamics and singularity theory. Thom had in mind applications to Biology.

Takens and myself proved that the stable one-parameter families of gradients are open and dense. The work was published in the Annals of Mathematics, vol. 118.

It was very rewarding to see the power of the geometric method, which I introduced in my PhD thesis some fifteen years before, as a main ingredient in this much more sophisticated question.

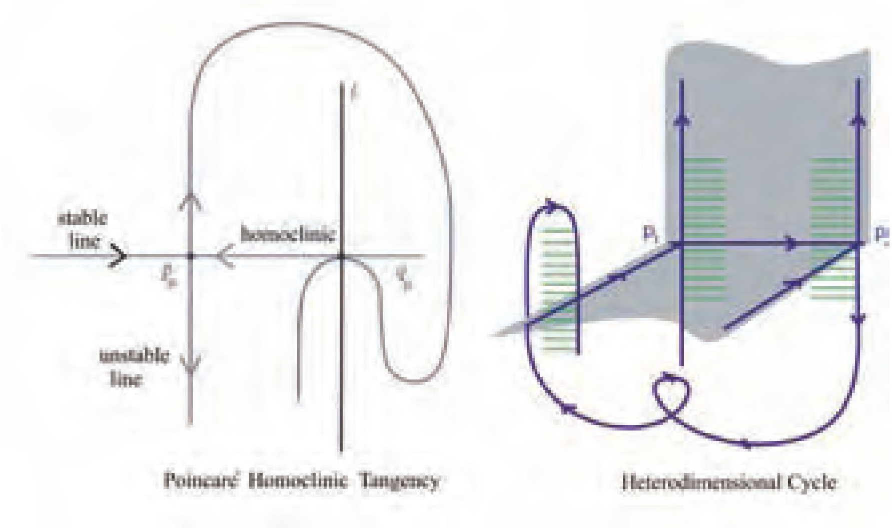
Some years later, in a work mixing techniques of my previous work with Takens and singularity theory, Dias Carneiro and myself proved the same result for two-parameter families of gradients: the stable ones are open and dense. It was a long paper published in Publications Mathématiques de l'IHES, vol. 70, one of the finest journals in mathematics.

In the middle of the 1980s, I started my collaboration with Jean-Christophe Yoccoz, a very talented young mathematician. In a series of articles that appeared in the Annales Scientifiques de l'Ecole Normale Supérieure, vol. 116, and the Bulletin of the Brazilian Mathematical Society, vol. 2, new series, we solved the centralizer problem for open and dense subsets of many classes of hyperbolic dynamical systems, that is, they commute only with their own powers.

Yoccoz was awarded the Fields Medal in 1994. In the laudation ceremony where he was presented with the award, our joint work on centralizers was explicitly mentioned, as well as our work on homoclinic bifurcations, to be briefly described below.

I turn now to another fundamental phenomenon for parametrized families of dynam-

ics: the unfolding of homoclinic tangencies. As Poincaré predicted, this is at the heart of the difficult problems in dynamics. In recent decades, after the remarkable work of Newhouse on the existence of infinitely many simultaneous sinks, we have improved our knowledge very much of what such a bifurcation may impose on the dynamics.
I believe that I have substantially contributed to that, either directly or by asking the right questions to be considered. In particular, I explicitly put homoclinic tangencies as well as heterodimensional cycles at the centre of a global understanding of dynamics: the description of the typical orbit of a typical dynamical system.
In a sense, I have not forgotten Poincaré's sentence that I have quoted above.



In fact, in my previously mentioned works with Newhouse or Takens or both, we have dealt with homoclinic bifurcations. Now, I wish to focus on how often a hyperbolic system undergoing a homoclinic bifurcation remains hyperbolic very near the unfolding parameter value, at least on surfaces. (Here, to simplify the language, I am calling hyperbolic a system whose nonwandering set is hyperbolic, with a dense subset of periodic orbits, and satisfying the strong transversality condition.)
The answer depends on the Hausdorff dimension of the system, which is the sum of the Hausdorff dimensions (fractal dimensions) of the stable and unstable foliations. If it is smaller than one, we have total prevalence of hyperbolicity. A weak version of this fact appeared in a previous work with Newhouse. The final version, including

how a homoclinic tangency can be created, appeared in two works by myself and Takens. The first one in Inventiones Mathematicae, vol. 82, and the second in the Annals of Mathematics, vol. 125.

In fact, I have conjectured that this result would not be true if the Hausdorff dimension as above is bigger than one. So, one side of the conjecture was still missing.

A strong indication that my conjecture was true was provided in my long paper with Yoccoz, “Homoclinic Tangencies for Hyperbolic Sets of Large Hausdorff Dimension”, published in the famous journal Acta Mathematica, vol. 172. An even stronger indication was produced by Yoccoz and myself in the paper “On the Arithmetic Sum of Regular Cantor Sets”, published in the Annales de l’Institut Henri Poincaré, Analyse Non Linéaire, vol. 14.

The question was finally solved by Moreira, one of my former students, and Yoccoz, for hyperbolic systems on surfaces, in a wonderful work published in the Annals of Mathematics, vol. 154.

Moreira, Viana and myself are now proving that the conjecture is indeed true in general, that is, in all dimensions.

At this point, as promised before, we shall mention the work by Newhouse on the unfolding of homoclinic tangencies on surfaces to show that, among C^2 diffeomorphisms, the hyperbolic ones are not dense. So, in other words, the hyperbolic diffeomorphisms are not typical, once again disproving Smale’s global conjecture. He had done so, showing that close to the bifurcating parameter value, there are, in the parameter line, small intervals with the following property: in any of them there is a dense set of points for which the corresponding diffeomorphisms display infinitely many simultaneous attractors (sinks). This fact contradicts hyperbolicity.

Viana and myself proved that the same is true in all dimensions. This work “High Dimension Diffeomorphisms Displaying Infinitely Many Periodic Attractors” was published in the Annals of Mathematics, vol. 140.

The work we carried out on the unfolding of homoclinic tangencies, as well as Smale’s on horseshoes arising from transversal homoclinic orbits, up to 1993, appeared in a book by myself and Takens, published by Cambridge University Press, called *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Again, this book had large repercussions. Indeed, it was at the time considered the definitive work on the subject.

I must say that this was not to be the last word, since just a few months ago the very prestigious journal Publications Mathématiques de l’IHES dedicated a whole issue, 217 pages, volume 110, to a work done by Yoccoz and myself [PY], entitled “Non-Uniformly Hyperbolic Horseshoes Arising from Bifurcations of Poincaré Heteroclinic Cycles”. In it, we deal with one-parameter families of surface diffeomorphisms that are initially hyperbolic and then go through a heteroclinic tangency. We expected all the

results in the paper to also be true, substituting heteroclinic by homoclinic tangency. Moreover, we believe that the several mathematical structures created and developed in the paper should be of much use in the solution of other, perhaps many, questions in dynamics. We show, among other facts, that even if the Hausdorff dimension of the diffeomorphisms in the family just prior to the bifurcating parameter is bigger than one, but not much bigger, then, we have prevalence of no-attractor, although we have no prevalence of hyperbolicity. Indeed, I have conjectured that the set of parameter values corresponding to infinitely many coexisting attractors shall have zero measure (Lebesgue).
That was a key point in my proposal, often referred to as the Palis global conjecture or programme: a description of the "typical" behaviour of trajectories for "typical" dynamical systems. This I will now present [PI], [PII]:

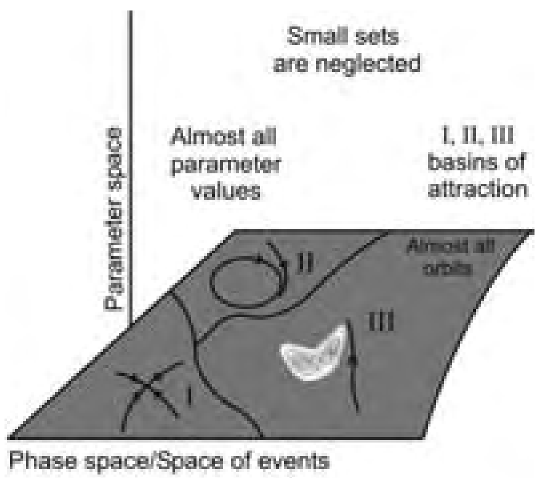
Main Conjecture

(A1) Every system can be C^r, $r \geq 1$, approximated by one displaying only finitely many attractors. Such attractors should support a Sinai-Ruelle-Bowen (SRB) invariant measure and the union of their basins of attraction should be of total probability (Lebesgue) in the phase space.

(A2) The attractors are stochastically stable.

(A3) For generic (non-degenerate) parametrized families with finitely many parameters, the systems with the properties presented above have total probability (Lebesgue) in the parameter space.

We notice that, if the Main Conjecture is true, it would be possible to convey a precise sense of uncertainty for a typical dynamics. Indeed, a typical dynamics would display only a finite number of attractors and the uncertainty of the future behaviour of its trajectories would be measured by the maximal value of the diameters of the attractors.



Parameter space × Phase space

A Supporting Conjecture
Any dynamical system can be C^r, $r \geq 1$, approximated by a hyperbolic one with the no-cycle property or by one exhibiting homoclinic tangencies or heterodimensional cycles.
There is a similar Main Conjecture for flows. The corresponding supporting conjecture requires the notion of singular cycles: they are cycles involving at least one singularity. Then the supporting conjecture would state:

Supporting Conjecture for Flows
In any dimension, every flow can be C^r, $r \geq 1$, can be approximated by a hyperbolic one or by one displaying a homoclinic tangency or a singular cycle or a heterodimensional cycle.

Some Basic References:

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[PY] J. Palis and J.-C. Yoccoz, *Non-uniformly hyperbolic horseshoes arising from bifurcations of Poincaré heteroclinic cycles*, Publications Mathématiques de l'IHES No. **110**, 1–217, 2009.

Alberto Quadrio Curzio:
Thank you, Jacob Palis, for your precise presentation.
I would now like to invite Carlo Sbordone, Professor of Analytical Mathematics at the University of Naples Federico II and a member of the Accademia Nazionale dei Lincei, to respond to Professor Palis's presentation.

Comments, Questions and Preliminary Discussion

Carlo Sbordone:
I am honoured to be here, but it is not easy to be a discussant on a mathematical subject when there is a general audience and not a specialist one. However, I will try to adequately simplify the most difficult concepts. First I would like to say that we mathematicians are all impressed by your recently proposed comprehensive set of