Computing Three-Dimensional Fluid Motion

Dennis Sullivan

I'm going to use one small piece of mathematics that some of you may remember from secondary school, the "quadratic equation". You have an equation, "a" times "x squared" plus "b" times "x" plus "c" equals zero. You can solve it with the nice quadratic formula which evolved over the millennia and flourished and was extended to cubic equations in Renaissance Italy, with Pacioli, dal Ferro, Tartaglia and Cardano in the 1500s.

Fluids are described by a more complicated quadratic equation, but there is some similarity in form, so let me discuss the simpler equation. When you solve this quadratic equation, there is a good region and a bad region. We insist on real number solutions in our analogy, since fluids are real. We cannot use imaginary numbers directly to describe fluids; we have to use real numbers.

In the good region, there would typically be two real numbers that solve our quadratic equation. In the bad region there are no real solutions. Let's imagine that our equation depends on time. For example, one or more of the coefficients a, b or c of the pieces in the equation depend on time, and let us assume that we start off in the good region. Then these two real numbers are moving around as time varies. What happens? They could actually move around and then come close together, touch each other and bounce off, and everything would be fine. Or they could come together, and – poof! – disappear. The quadratic equation has passed into the bad region.

You may also recall from school that there is a certain combination of the elements in the equation that can keep you safely in the good region. One knows if the time evolution keeps the quantity "b squared minus four times the product of a and c" from becoming negative, then one stays in the good region with real solutions. If only we had such a discrimination in the more complicated quadratic fluid equation, we could know when and why the mathematical fluid solutions would exist for all time.

Now I would like to tell how I came to work on the fluid problem in the early '90s and to present some of the ideas that have emerged since then. All of this has led up to the Balzan project, which will combine these elements.

Serious mathematics for me started by studying topology, even though before that I was more interested in things related to differential equations, which are what the quadratic equation related to fluids is about. In graduate school at Princeton in the 60's, topology was very much in a golden age. There was, for example, a certain theorem that we heard about as new graduate students. Suppose you have a smooth solid ball of earth, and you have very fine grass growing on the ball. If you try to comb the grass flat, there is always some place where the grass will stick up. This is a special case of a vast generalization of this essentially topological statement.

In any space – under very general circumstances – when you have any ordinary differential equation, which means you have a direction in which you want to move specified at each point, there is a certain very pertinent topological invariant, an integer number called the Euler charactersitic. If this number is non-zero (and it is two for the smooth surface of the ball), then for any ordinary differential equation on this space, there always has to be a place where the direction field has to have a singularity. You cannot lay the direction field flat. I was very impressed by this, because it solved a general problem. It was a simple statement. It gave a criterion to decide whether or not you could have a differential equation without singularities on a given space.

Thus, at Princeton one was easily attracted to study topology, which has a very geometric origin but uses the power of algebra to study the geometry of space. The field of algebraic topology was developed quite far during that golden age at Princeton based especially on the ideas of Norman Steenrod. One could compute in algebra and make deductions about geometry. The above mentioned theorem, the Euler Characteristic Theorem, seemed fantastic, because it showed how this algebraic invariant completely solved that particular problem about singularities in the theory of ordinary differential equations. So I pursued this type of thinking for about a decade, studying algebraic topology and developing some algebra related to differential forms which influenced and became part of what is now called homotopical algebra.

Topology studies a space by breaking it up into little boxes, or cubes in the case of three-dimensional spaces. Then it studies how the different pieces fit together, making an algebraic structure out of the linkage between the different pieces. The advantage of this over the idea of Newton, which is the idea of the infinitesimal or of calculus, is that here in this combinatorial world of algebraic topology you have only finitely many entities in a combinatorial relationship or in an algebraic relationship, and with this algebra one finds out rather deep properties of the space. So topology is determined by these configurations of little cubes, and it does not matter how you divide the space into cubes. You squeeze out information from any decomposition into cubes. It is a powerful method. The disadvantage, which will be discussed below, is that this finite picture enjoys less symmetry than the picture of Newton. This symmetry does not disappear completely, but is remembered approximately using Steenrod's ideas.

One might now ask: where did this idea of algebraic topology come from? Here is one part of the story. In following Newton's ideas on the motion of the planets, it turns out you can solve exactly, as Kepler had foreshadowed, certain simple problems. However, as soon as you add one more body to the solvable configurations, the Newton equations are too complicated to solve explicitly. Poincaré invented the topic of qualitative dynamical systems to help discern patterns in these difficult-to-solve situations. Simultaneously and in parallel, Poincaré introduced topology, both algebraic and geometric. Many who were near to my generation around that time in the 70's became involved in topology, pictures and dynamical systems, and further tools were developed and absorbed.

One day at the Institut des Hautes Études (IHES) in Paris, which is interdisciplinary between mathematics and physics, I wandered into a lecture in physics. After a great deal of discussion about harmonics, it became clear that the seminar was studying fluids by equations and formulae. I was astonished because they had the equations that described how fluids moved around in space, and they could start trying to calculate. But because of Newton's infinitesimals, arbitrarily high frequency harmonics had to be utilized, or in other terms, they had to consider not only cubes of a fixed size, but in principle had to go down to cubes of an infinitesimal size. There were infinitely many things one had to compute. The answers might not converge. They might not make sense. They might not approximate anything. I thought to myself, "What? This is a beautiful equation and it describes fluids, but one doesn't know whether the solutions can be computed? One doesn't know whether solutions exist?" I was astonished because these equations are used in the aircraft industry to build airplanes; engineers use them all over the world to model everything. Fluids exist; they know what to do. We have a beautiful equation, and yet the basic foundation – the mathematics, or the epistemology of this equation – is very sparse. If you start with a smooth initial condition, the fluid evolves for a small amount of time, depending on that initial condition. Then it may start to get more violent, the time of predictability or control gets shorter and shorter, and then maybe – like those two real zeros of the quadratic equation – they come together, and – poof! – the solution disappears.

I needed a break from the end game and from writing about results on the Feigenbaum renormalization conjectures I had been working on for nearly a decade. I wanted to try to find out why that three-dimensional fluid problem was so hard. I was not so presumptuous as to assume I could realistically work on this problem per se because I was not in that branch of mathematics, and I knew prodigious experts in that domain who had not been able to solve it. But one could still ask, "Why is this problem so difficult? Why is it not solved? Why can't it be solved?" Fluids are very real and this model is beautiful and it goes back for centuries. It's the right model from the Newton calculus point of view. What is going on? Maybe one can modify the question to a more tractable one.

It turns out that it's not difficult to find out what is going on. It turns out that for this equation, the amount of information that the equation gives is of two types. One is that the total energy is controlled, and you can use that. The other is local. The volume density and the twisting or vorticity is preserved. Fortunately, energy is a number. You have so much energy to spend, like a budget. That is all you have; the fluid has to get by with that amount. Volume density is also a number at each point. But vorticity is not a number; it is what we call a tensor. It has a direction and a magnitude at every point, and it is preserved in the sense that the fluid moves this tensor around, transforming it appropriately, and in that transformed sense, vorticity is preserved.

Now, this is the picture in three-dimensional space. If the fluid is restricted to a surface [the atmosphere in some respects fits this description], vorticity measures locally the spinning in a plane tangent to the surface, the vorticity direction is pointing out of the surface at all points. So it is not really a complicated tensor, but rather a number, the amount of twisting at each point. In fact, in two dimensions, you have two preserved scalar quantities: area density and scalar vorticity. The foundational theory of two-dimensional fluid equations is completely and beautifully worked out based on the preservation of these two scalars. The mathematics is perfect. The problem in three dimensions, then, is that somehow the preserved quantities are more elaborate and the methodology of partial differential equations has not yet made use of the tensorial preserved quantity to control the solutions of three-dimensional fluid motion.

In fact, in a recent paper and blog, the famous mathematician Terence Tao has actually produced an evolution equation similar to the fluid equation with friction or viscosity which has exactly the same energy and volume qualities. But by making the different scales interact in a very technically complicated way, Tao actually gets solutions of this equation to blow up in finite time. The role of vorticity is altered by the friction.

So we learn one idea from the success in surface fluid flow. We should focus more on vorticity in three-dimensional fluid flow. Vorticity has always been geometrically attractive (as in Arnold's work) even though it is preserved only in this more complicated tensorial sense. We also can use common sense and pursue a practical idea. In dimension two, where the mathematical theory is quite complete, calculating fluids is still not a completely successful endeavor. So the second idea is to emphasize the vorticity version of two dimensional fluid equations coupled with Steenrod's considerations to create computational algorithms that are successful. This makes sense, because we know that what we are trying to compute actually exists in the Newtonian mathematical model. If we can do this, we will use the same ideas to invent algorithms in dimension three that may help compute fluids practically, whether or not the desired foundational mathematics is actually true in three dimensions.

Let's discuss the methodology in the practical second idea. This comes from algebraic topology and it directly addresses the non-linear or quadratic term in the equation via Steenrod's hierarchy of homotopies. In this discussion one can start, as is widely understood, to discretize the equations using linear principles from algebraic topology (finite differences, averages and the relationship between bulk and boundary integrals). One obtains algorithms that may depend on how the continuum equation is written and discretized.

Let me explain. If you have the Newton continuum model for the fluids, you can write the continuum equations in many different ways, because Newton's calculus has all sorts of beautiful algebraic properties: commutative multiplication, associative multiplication, certain formulae for differentiation and so on. Then you can take one of these ways and give it to an applied mathematician or a scientist engineer, who may attempt to put it on a grid and discretize it.

We believe now that the different ways of writing continuum equations and then discretizing them to first order or second order in different continuum formats may well lead to rather different algorithms. However, we also believe that if we continue the discretization at one scale incorporating the Steenrod hierarchy of corrections to the breakdown in the discrete world of the algebraic symmetry of the continuum world, we obtain discretization schemes that though quite different at low orders become essentially equivalent when the entire hierarchy is used. We may be finding a conceptual approach to taking a non-linear problem, discretizing it into algorithms and understanding mathematically the different discretizations.

The key idea is to use a Steenrod type hierarchy of small deformations that corrects the necessary breaking of algebraic symmetry in discretization processes applied to nonlinear problems expressed in the continuum language. There is more I can say here.

Suppose actual solutions to the actual fluid evolution equations expressed in terms of differential forms are encoded into these hierarchical models in a manner consistent with the principles of homotopical algebra alluded to above. A colleague at CUNY, John Terilla, has recently communicated a very simple and elegant way to do this. It follows that the measurements of the real solution as tabulated by integrals over pieces of the cubical decomposition of the hierarchical models can also be computed from solutions to the combinatorial evolution models defined by the models. The proof is based on the principles of homotopy invariance from algebraic topology as manifested in homotopical algebra.

The idea of the Balzan Project is to take these topology ideas, apply them to the non-linear three dimensional fluid equation using algorithms with the hierarchy of corrections. Young researchers, with their knowledge of computer programming, can thus use the resources of the project to actually test the various theoretical algorithms that can be generated. They can get answers that are correct when the continuum model of fluids has solutions. These can be checked against empirical data. We expect the computed answers will thus be meaningful and useful even while the level of theoretical certainty about the continuum model for fluids in three dimensions remains rather sparse.

To close, I report that there is one class of finite dimensional models (not yet enriched with the Steenrod hierarchy of corrections), which has all of the properties of the continuum fluid model in terms of preserved quantities, including vorticity. Moreover, this class of models is characterized by these invariance properties. This class has one further property expressible and provable using finite dimensionality. The new property is the following: there is a Gaussian type statistics or measure on the set of fluid states, which is invariant under the evolution in the model. This leads to a conjecture that makes sense in infinite dimensions for the continuum model. It is that for almost all initial conditions relative to the limiting Gaussian measure, the solutions to the 3D fluid equations can be uniquely defined for all time.

Thank you.